A NUMERICAL PROCEDURE FOR TWO DIMENSIONAL HEATING AND MELTING CALCULATIONS, WITH APPLICATIONS TO LASER EFFECTS

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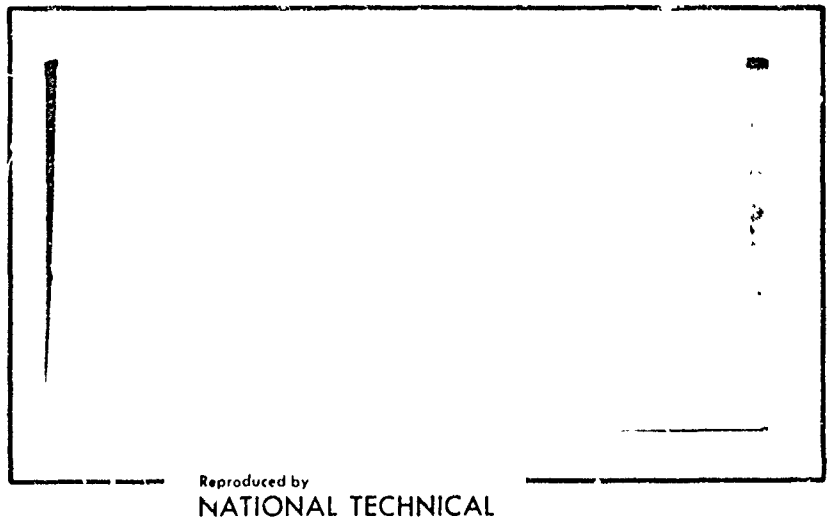

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FIGURES

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A method for determining the two dimensional transient temperature distribution and progression of melting in disks subjected to an applied flux over one face is given. The calculations are performed by dividing the solid into a number of finite elements and performing a heat balance over finite time increments. The method was found to give good agreement with known solutions for two dimensional heat conduction problems and one dimensional melting problems. Two dimensional melting problems in alumina, titanium, stainless steel and magnesium are also considered as examples of the method. The results include a demonstration that the time required to melt through approaches the time predicted by a one-dimensional (axial) heat balance if the power per unit thickness is sufficiently large.

I. INTRODUCTION

If the radiation from a high intensity laser is directed onto the surface of a solid, significant temperature rises will occur, and will eventually, in the case of a continuous incident flux, lead to melting and possibly vaporization of the material at the surface. As time progresses, the region of melting may be expected to increase both radially from the beam center, and into the solid. The heating of the material will also lead to thermal expansion which will, in turn, give rise to thermal stresses which may be of sufficient magnitude to cause fracture. This phenomenon is most likely to occur in materials of limited ductility, as ceramics; however, in a composite material, or in a composite structure, the differing coefficients of thermal expansion may give rise to internal forces which bring about failure of the bonds between the constituents, leading to significant degradation of the strength and usefulness of the part or structure.

In addition to melting, vaporization, and fracture due to thermal stress, damage may also be brought about by other means. If the heated object is a load carrying member, the interaction of the thermal effect with the load must be considered. The increase in temperature may cause such a reduction in the yield strength of the material that the load carrying ability of the part or structure may be seriously reduced, causing failure to occur at a load which would be in the safe range for the unheated structure. Creep rates may also be higher at the elevated temperatures brought about by laser heating. Damage may also be caused in an indirect manner, as for example, if the laser heating brings about the removal of a protective coating.

In view of these many possible consequences of heating of solids, it is imperative that methods for determining the resulting temperature distributions be available. For a material which is thermally isotropic and homogeneous, so that conductivity, density and specific heat are each a single constant, the temperature, as a function of time and position, is the solution of the familiar heat conduction equation. If, in addition, all thermal properties are independent of temperature, the equation is linear with constant coefficients. Solutions for some interesting and important problems may be obtained in closed form, or in terms of a Fourier series. Many such solutions have been collected by Carslaw and Jaeger [1]. On the other hand, if the thermal properties vary from point to point, or are functions of temperature, the possibility of finding such solutions becomes very unlikely.

Problems for which an exact solution cannot be obtained may be attacked by approximate or numerical methods. Finite difference methods have been used extensively, and have been applied to both one and two dimensional problems [2-6]. In addition, finite element methods for heat conduction calculations have been developed in the past decade [7-9]. Surveys of recent work are also available [10,11]. One disadvantage of the numerical method is that thermal stresses cannot be determined easily with good accuracy, for the thermal stresses depend on the gradients of temperatures, and these cannot be determined with good accuracy by a method designed to yield numerical values for the temperatures themselves. A variational method for heat transfer problems has been given by Biot [12] and has been applied to problems including phase changes [13,14]. A closely related approximate method has been given by Goodman [15,16]. To date,

these methods have been applied to only one dimensional problems, but could as well be used in multi-dimensional heat conduction.

The phenomenon of phase change is not described by the heat conduction equation, per se, but problems involving a change in phase have been treated by introducing a moving boundary separating the two phases, and attempting to solve the heat conduction equation in each phase, with the appropriate matching being enforced at the interface. Problems involving such a moving boundary are inherently non-linear, and plane one dimensional problems have been treated most extensively [3, 13, 14, 15, 16] by the above mentioned methods, and others [17-20]. In this case, the moving boundary is plane. In two or three dimensional heat conduction, both the location and the shape of the boundary are unknown, leading to much more difficult problems [21,22]. Multiple phase changes have also been considered [23].

In real materials, the thermal properties are functions of temperature, and it is unlikely that any exact solutions will be found, although the approximate methods can be applied [24]. The possibility of finding exact, or even good approximate solutions, for heat conduction problems in composite or layered materials is quite unlikely. Such problems can be more successfully treated by one of the numerical methods.

In applying the method of finite differences one first writes down the governing differential equation, and then seeks to develop a set of difference, or algebraic equations through the process of discretization. In the finite element method, as used elsewhere, [9], a simple, approximate temperature distribution is assumed for each finite element, and defined in terms of nodal temperatures. The resulting temperature field is substituted into a global energy functional, the first variation of which

leads to a set of algebraic relationships between the nodal temperatures, at each time. In a transient problem, this system of equations must be solved at each time step, necessitating the inversion of a large matrix at each time increment. A finite element approach is expected to be particularly advantageous for heat conduction in an inhomogeneous material, whether the inhomogenieities are initially present, or arise from temperature dependent properties, for the thermal properties of each element need not be the same, or constant. The possibility of treating a phase change is also apparent, through allowing certain of the finite elements to be of one phase, while other elements are of another phase. In such an approach, however, one would wish to use extremely small elements so that the phase boundary can be located with good accuracy. In the finite element methods previously developed, the goal has been to use as few elements as possible, because of computer storage limitations and the long running times required to solve a large system of equations at each time increment. Consequently, in the present work, a different finite element approach was used, one which does not require the solution of a large system of equations at each time increment. For the same number of elements, the present method is therefore expected to be less accurate than the traditional finite element methods, but is expected to be more feasible for the approximate determination of the liquid-solid interface in two dimensional heat conduction problems with change of phase.

In the section to follow, a general method for solving heat conduction problems with prescribed boundary flux will be given. The geometry to be considered is of practical interest, namely, a plate, slab, or sheet of uniform thickness, subject to a prescribed flux on one face. In the third

section, some comparisons between results obtained with the new method are compared with results obtained by other methods, both exact and approximate. Finally, in Section IV, some results for two-dimensional heat flow with change of phase are given.

Although numerous, the works cited here by no means give a complete survey of the methods available for heat conduction problems, nor of results obtained for problems involving a change of phase. For a more complete treatment (through 1964), the review by Muehlbauer and Sunderland [25] may be consulted. More recent work is described by Boley [26].

II. DEVELOPMENT OF FINITE ELEMENT METHOD

We consider a cylindrical disk, with a prescribed flux having a known variation in the radial direction and constant in time applied over the central region. Exact solutions to this problem have been given for temperatures below melting in a homogeneous and thermally isotropic disk [27]. For sufficiently large flux, the temperatures within the disk will be brought to the melting point. The determination of the temperature and the location of the solid-liquid interface then becomes quite difficult, and has been extensively considered only for one dimensional problem, such as occur when an infinite slab is subject to a uniform flux.

The approach to be taken is to divide the region of interest into a large number of finite segments and perform a heat balance on each. We begin by dividing the thickness, ℓ, of the disk into a number, N_R, of layers which, for convenience, will all be taken to be of the same thickness, although each layer can easily be assigned a different thickness, if desired. In a similar manner, the radius, a, is divided into N_C equal segments, although unequal segments may again be used, if desired. For the case of an axially symmetric flux, there will be no circumferential flow of heat, nor any circumferential temperature variations, and consequently, no subdivision in the circumferential direction is required. Hence, the disk has been divided into $N_C \times N_R$ segments, each of which is in the form of a solid annulus. In order to provide for eventual inhomogeneities in the thermal properties which may arise either through consideration of a composite material or as a consequence of material properties which are a function of temperature, or because of a phase change, it is found to be convenient to define certain arrays ($N_R \times N_C$) describing the properties

of each finite element. In all that is to follow, the first index denotes the row number, as in Figure 1, and is related to the z or thickness coordinate, and the second index denotes the column number, and is related to the radial or r coordinate. The array shown in Figure 1 may be thought of as the right half of a diametral section of the disk.

Let

r_{I_j} = inner radius of cells in column j, cm.

r_{O_j} = outer radius of cells in column j, cm.

A_{Z_j} = area of interface separating cells in column j
$= \pi(r_{O_j}^2 - r_{I_j}^2)$, cm^2.

A_{R_j} = Area of interface separating cells in column j from cells in column j + 1
$= 2\pi r_{O_j} \ell/N_R$, cm^2

ρ_{ij} = density of cell of row number i and column j, gm/cm^3

M_{ij} = mass of cell i, j, gm.
$= \rho_{ij} A_{Z_j} \ell/N_R$

L_{ij} = heat of fusion for cell i, j, joules/gm

$K_{R_{ij}}$, $K_{Z_{ij}}$ = conductivity of cell i, j in radial and axial directions respectively, joules/(sec cm °C)

$Q_{R_{ij}}$, $Q_{Z_{ij}}$ = rates of heat flow out of cell i, j in radial and axial directions, respectively, joules/sec.

θ_{ij} = Average temperature in cell i, j, °C

$C_{p_{ij}}$ = specific heat of cell i, j, joule/(gm°C).

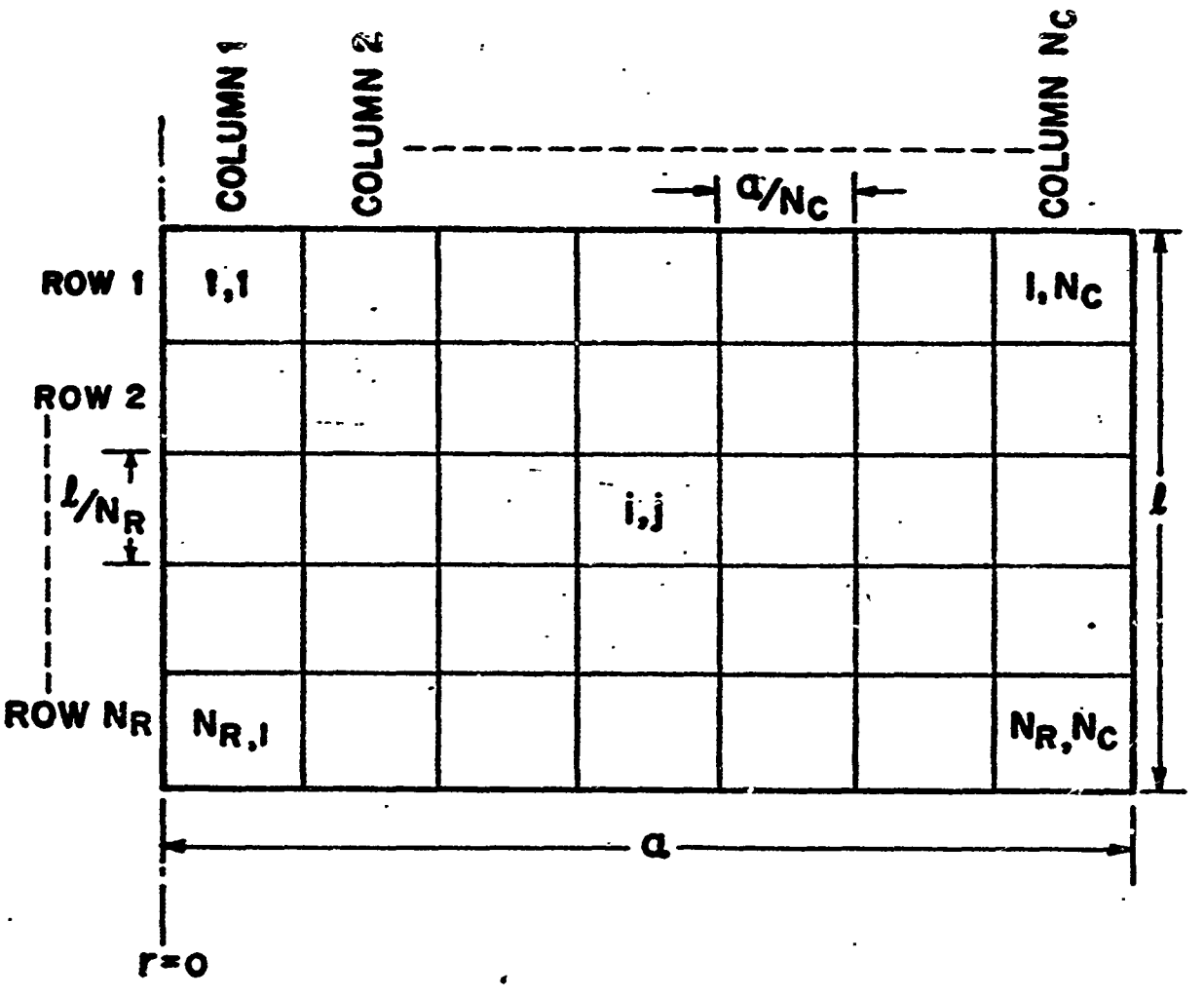

Figure 1 Division of Disk into Finite Elements

The heat fluxes for a typical element are shown graphically in Figure 2. The rate at which heat flows radially from cell i, j into cell i, j+1 is given approximately by the product of the conductivity, the area of the interface, and an approximate value for the temperature gradient at the interface between these two cells, or

$$Q_{R_{ij}} = K_{R_{ij}} A_{R_j} \left\{ \frac{\theta_{i,j} - \theta_{i,j+1}}{a/N_C} \right\} \tag{1}$$

Similarly, the rate at which heat flows from the cell i, j axially into the cell i+1, j is approximately

$$Q_{Z_{ij}} = K_{Z_{ij}} A_{Z_j} \left\{ \frac{\theta_{i,j} - \theta_{i+1,j}}{\ell/N_R} \right\} \tag{2}$$

In addition to the flow of heat from one cell to another through conduction, the heat balance must take into account the possibility of heat addition to each cell from external sources, and from internal reactions, through a heat addition term for each cell which includes both effects.

$$F_{ij} = F_{E_{ij}} + F_{I_{ij}} \tag{3}$$

F_{ij}, the heat added, has dimensions of joules/sec. The heat balance is performed for each cell by adding the heat entering cell i, j from the neighboring cells i-1, j and i, j-1 to the flux added, F_{ij}, and subtracting the flux leaving, $Q_{R_{ij}}$ and $Q_{Z_{ij}}$. If cell i, j is at a temperature less than the phase transformation temperature, the temperature will rise in a time Δt an amount given by

$$\Delta\theta_{ij} = \Delta Q_{ij} \cdot \Delta t/(C_{p_{ij}} M_{i_j}) \tag{4}$$

$$\Delta Q_{ij} = F_{ij} + Q_{R_{i,j-1}} + Q_{Z_{i-1,j}} - Q_{R_{i,j}} - Q_{Z_{i,j}} \tag{5}$$

The temperature rise will continue in later time increments Δt until such time as the temperature at which the phase transformation takes

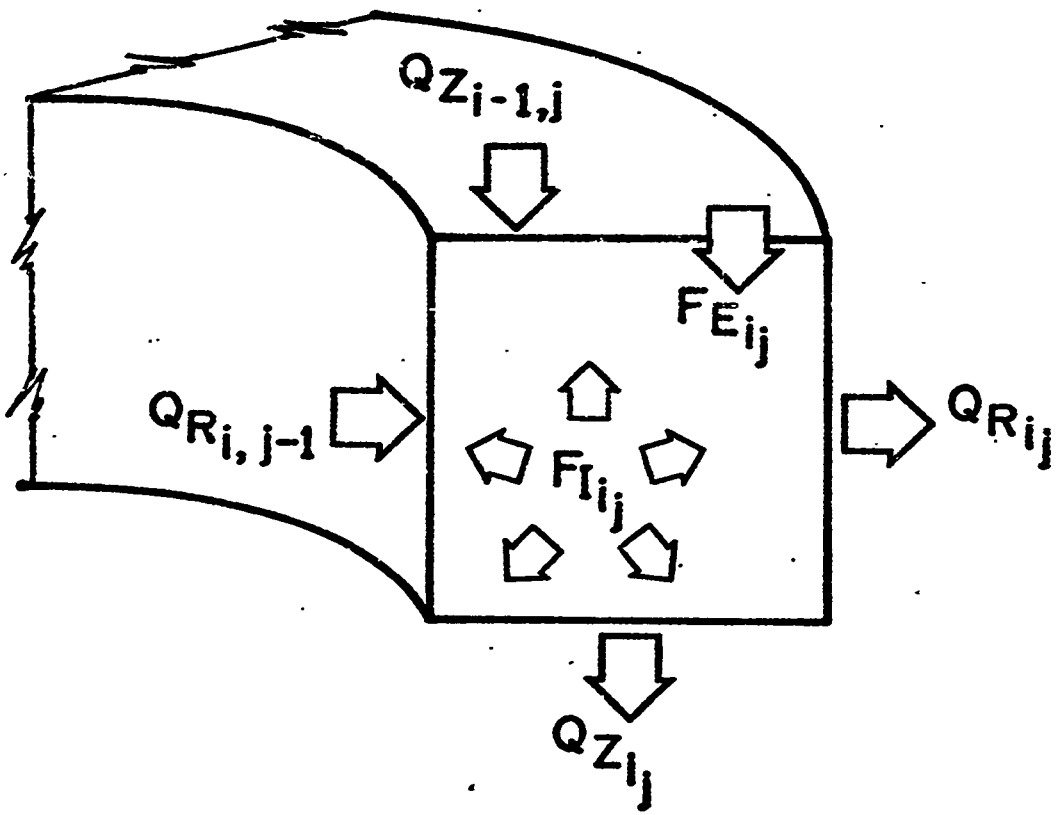

Figure 2 Heat Balance on Typical Finite Element

place is reached. The total energy which must be added after the melting temperature is reached in order to bring about a complete change of phase of all the mass in cell i, j is given by $L_{ij}M_{ij}$. It is convenient to compute this quantity for each cell, and store the result as an array P_{ij}. Then, beginning with the first time increment in which the temperature of the cell i, j is at or above the melting temperature T_m, the stored quantity P_{ij} is reduced from its current value by the amount $\Delta Q_{ij}\Delta t$ in the time increment Δt and the temperature is held fixed at the melting temperature. At K time increments after cell i, j first reaches T_m,

$$P_{ij} = P_{ij} - \sum_{k=1}^{K} \Delta Q_{ij_k} \Delta t_k \qquad (6)$$

and the entire cell is presumed to complete the phase change during that time increment in which $P_{ij} \to o$. The material in cell i, j is then permitted to increase in temperature once again, with specific heat and conductivity now being appropriate for the new phase, if it is assumed to remain on the surface. Alternatively, the material which has undergone the phase change may be assumed to be instantaneously removed, as in the case of ablation, or melting where sufficient forces are present to overcome the surface tension of the melt and remove the melted material as it is formed. In the case of complete melt removal (or equivalently, ablation), as is considered in all of the examples to be discussed in the next sections, the conductivities, $K_{R_{ij}}$ and $K_{Z_{ij}}$ are set to zero at the time the material is removed, which has the effect of removing the cell i, j from the heat balance calculations. The flux $F_{E_{ij}}$, representing the flux added externally to cell i, j is transferred to the cell immediately below, i.e., to cell i+1, j at the same time.

The heat balance for cells or elements on the boundary must be handled somewhat differently from those on the interior so as to provide for the approximate satisfaction of any given boundary conditions. In the case of a prescribed boundary temperature, the temperature of that boundary cell is held fixed for all time increments. In the case of an insulated boundary, the net flux in the direction normal to the boundary is set to zero. In the case of a prescribed flux, the net flux in the direction normal to the boundary is set to the desired value, and in the case of a boundary condition described by Newton's law of cooling, the flux normal to the boundary is set to the difference between the temperature of the boundary cell and the surroundings, with the appropriate multiplicative constant included. By symmetry, the radial flux entering cells in column 1 is zero.

Because of the approximate way in which the conduction is computed (Equations 1 and 2) the time increment Δt must be sufficiently small in order to insure stability of the solution. The flux for the entire time increment is computed based on the temperature difference between adjacent cells at the beginning of the increment. It is evident that if the time increment is permitted to be too large, sufficient heat would flow to cause the initially colder element to become hotter than the initially hotter element, which is obviously impossible. A physical derivation of the appropriate stability condition is easily obtained. If two cells, the centers of which are a distance Δz apart are initially at temperatures T_1 and T_2, and if the conductivity is K, in time Δt the total heat flow will be

$$Q = KA\ \Delta t\ (T_1 - T_2)/\Delta z \qquad (7)$$

where A is the area of the annulus separating the two elements. If the specific heat (per unit volume) of each is ρC_p, then at the above flux, temperature equilibrium between the two elements is reached when

$$Q = \rho C_p V (T_1 - T_2)/2 \tag{8}$$

where V is the volume of each element, approximately $A\Delta z$ for each. Eliminating Q between Equations 7 and 8, we find

$$\Delta t = \rho C_p (\Delta z)^2/(2K) \tag{9}$$

But temperature equilibrium cannot be established in a finite time, hence Δt must be less than this bound. Moreover, in two dimensional heat flow, both components of flux must be considered, leading to a more stringent requirement. However, if the two element spacing distances, a/N_C and ℓ/N_R are not comparable, the maximum permissible time increment is satisfactorily determined by using the lesser distance increment in Equation 9.

A method given by Dusinberre [28] for the hand calculation of transient temperatures is very similar to the method used here to compute the heat conduction. If the thermal properties are uniform and constant, the method can also be shown to generate the same equations as would arise with the method of finite differences, with central differences being used in the space variables and a forward difference employed in time. The stability and convergence of such techniques applied to linear problems (constant properties and no melting) have been extensively studied [29]. Numerical methods for the treatment of the non-linear problems have also been discussed [30].

A computer program was written and used to perform the calculations described in the next section. A listing of that program, together with a brief documentation, is given as an appendix.

III. COMPARISONS WITH KNOWN SOLUTIONS

We will now consider several example problems, making comparisons between temperature fields predicted by the finite element method described in the preceeding section, and exact solutions, when possible, and other approximate methods, when available.

a. One Dimensional Heating of Thick Slab

As a first example, we will consider one dimensional heating of a thick (infinite) slab having properties approximating those of Al_2O_3. We take

$\rho = 3.8$ gm/cm^3

$K = 0.104$ joule/(cm sec °C)

$C_p = 0.885$ joule/(gm°C)

$T_m = 2313$°K

and consider a uniform flux of 4000 joules/(cm^{2}sec) striking material initially at 300°K. The exact solution, valid to the onset of melting, is known [1] to be

$$T - T_o = \frac{2F(\alpha t)^{1/2}}{K} \text{ ierfc} \left\{ \frac{x}{2(\alpha t)^{1/2}} \right\} \tag{10}$$

where α is the diffusivity, $K/(\rho C_p)$, and ierfc denotes the first integral of the complementary error function. The front surface temperature is given by

$$T - T_o = \frac{2F}{K} (\alpha t/\pi)^{1/2} \tag{11}$$

and, for the parameters given above, the surface melts at 0.0695 sec.

The temperature profiles given by Equation 10 are plotted at 0.02, 0.04, and 0.06 seconds in Figure 3 as the solid lines. Temperature profiles for the same heating conditions were determined by the finite element method through considering a slab 0.2 cm thick (from Figure 3, it

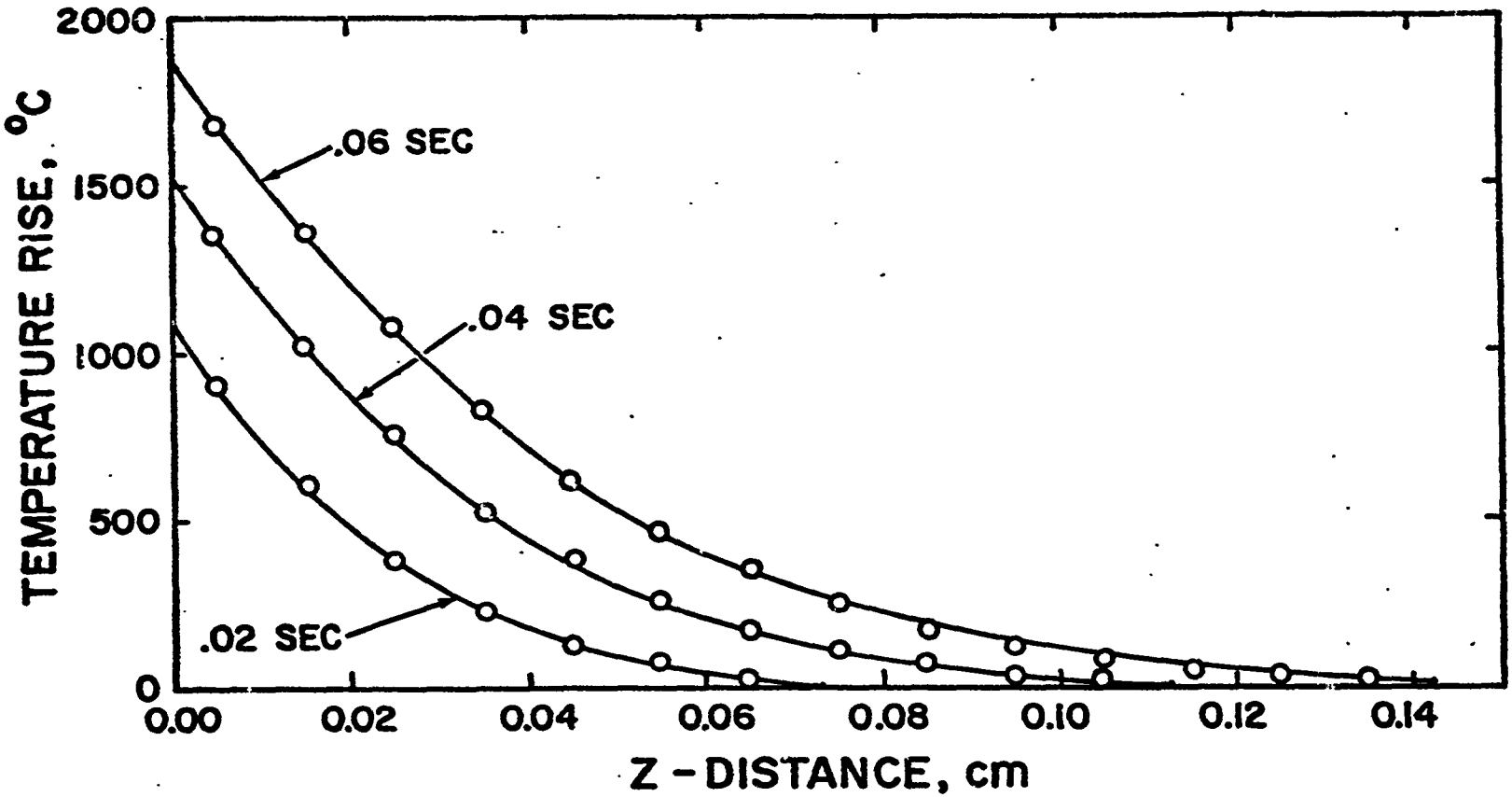

Figure 3 Temperature Distribution in Thick Slab of Al_2O_3, Exact and Approximate Solutions for One Dimensional Heat Flow, F = 4 KW/cm^2

can be seen that a slab of such thickness may be regarded as being of infinite thickness for times less than 0.06 seconds). The slab was divided into 20 and 40 layers, with time steps of 0.001 and 0.00025 seconds used in the two cases, respectively. (Time steps less than 0.0016 and 0.0004 are required here for stability). The average temperature in the first 0.01 cm of the material at t = 0.06 was found to differ by less than 0.2% between the two calculations. The results of the finite element calculations using the larger layer thickness and large time step are given as the circles in Figure 3. As can be seen, the results of the exact and approximate calculations are, for the most part, indistinguishable.

b. One Dimensional Melting of Finite Slab

As a second example, we consider the one-dimensional heating and change of phase of a finite slab, assuming the melt is instantaneously removed. For a prescribed uniform flux F on the surface $x = \ell$, and the other face ($x = 0$) insulated, the resulting one dimensional temperature field is [1]

$$T-T_o = \frac{F_E t}{\rho C_p \ell} + \frac{F_E \ell}{K}\left\{\frac{(3x^2-\ell^2)}{6\ell^2} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} e^{-\alpha n^2\pi^2 t/\ell^2}\cos\frac{n\pi x}{\ell}\right\} \quad (12a)$$

$$= \frac{2F_E\sqrt{\alpha t}}{K}\sum_{n=0}^{\infty}\left\{\text{ierfc}\left[\frac{(2n+1)\ell-x}{2\sqrt{\alpha t}}\right] + \text{ierfc}\left[\frac{(2n+1)\ell+x}{2\sqrt{\alpha t}}\right]\right\} \quad (12b)$$

valid until such time as melting begins at the front surface. Masters [3] has given results obtained by a finite difference calculation for the predicted recession of the front surface for an aluminum slab, using the properties

$T_m = 993°K$

$L = 418$ joules/gm

$K = 2.09$ joule/(cm sec °C)

$\alpha = K/(\rho C_p) = 1\ cm^2/sec$

$\ell = 0.3\ cm$

We assume the complete and instantaneous removal of the melt, resulting from the application of a flux of 41,800 joules/(cm^2sec). Taking the density to be 2.7 gm/cm^3, the value of specific heat required to yield a unit diffusivity is $C_p = 0.775$ Joule/(gm°C)

The total time required to melt a slab under one dimensional heating is readily determined from an overall heat balance to be

$$Ft = [L + C_p(T_m - T_o)]\rho\ell \qquad (13)$$

or 0.0186 seconds. From the exact solution, (Equation 12), the time required for the front surface to reach the melting solution was found to be 0.0011 sec., hence 0.0175 sec. are required to move the melting line from the front to the rear surface. This time was also computed by the finite element method, first by dividing the slab into 10 layers and using a time increment of 200 μ sec and by dividing the slab into 20 layers and using a time increment of 50 μ sec. The two melting times were determined to be 0.0182 sec and 0.0178 sec, indicating that the approximate method appears to be converging to the correct solution as the size of the increments is reduced. In both of these cases, the time steps used were just under 50% of the minimum time step required for stability. The location of the moving free surface is shown in Figure 4. The solid line is the result given by Masters [3], and implies a predicted time (> 0.02 sec) which is greater than the time deduced from the overall heat balance (0.0186 sec). The other two curves in Figure 4 depict the results of the finite element calculation, using 20 layers and $\Delta t = 50$ μ sec. The lower curve depicts the time at which various points first reach the melting temperature, and

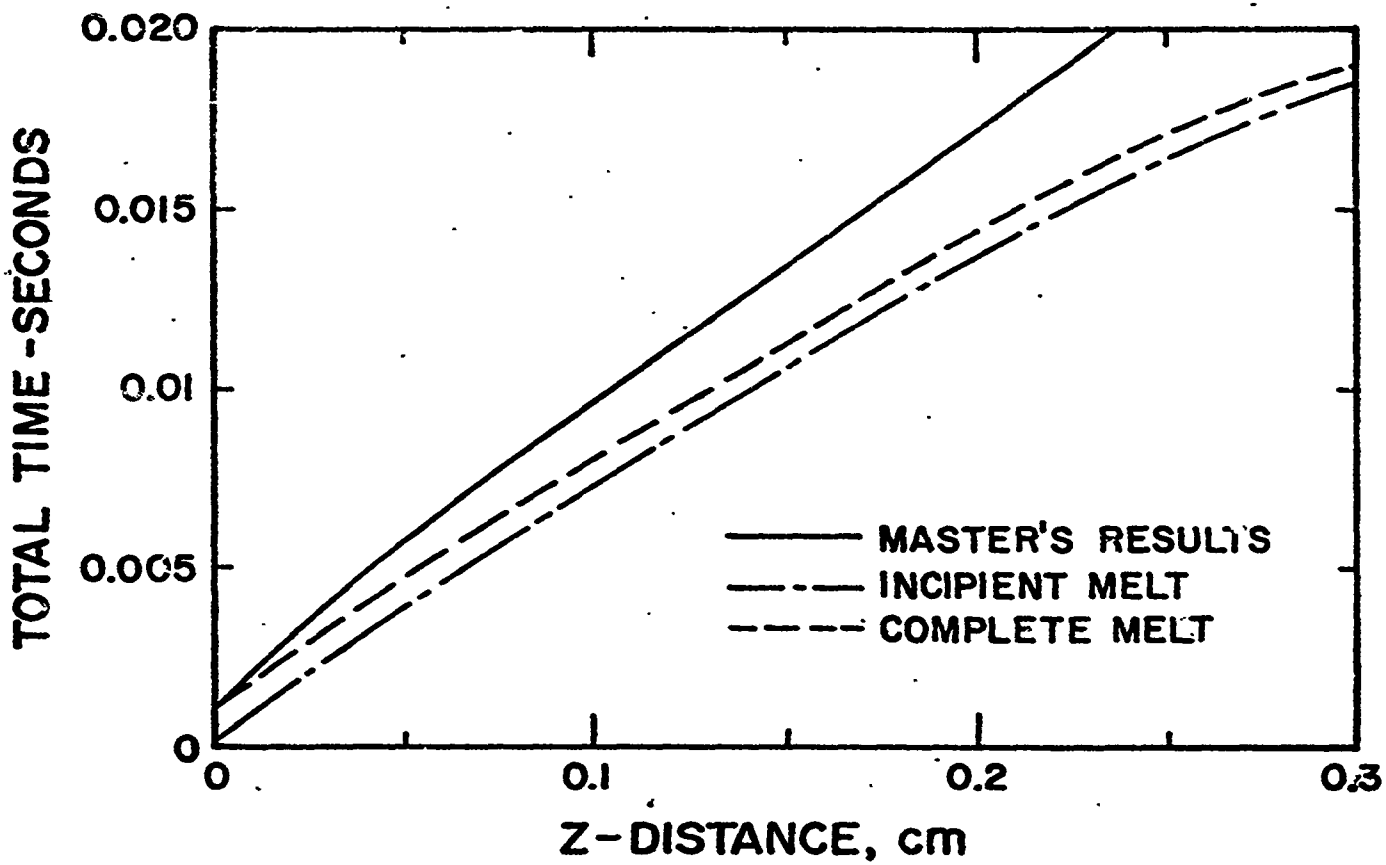

Figure 4 Free Surface Recession in 0.3 cm Aluminum Slab, One Dimensional Heat Flow, F = 41.8 KW/cm^2 by two Approximate Methods

the upper curve indicates the times at which particles are completely melted and removed. The interval between the curves indicates the region in the process of melting.

The problem of one dimensional melting with complete removal of the melted material has also been treated by Citron, who has given [14] an approximate (rather than numerical) method and some typical results. His results are presented in such a manner as to indicate that only a single dimensionless parameter, M/r, is required to characterize the problem.

$$\frac{M}{r} = \frac{\pi^{1/2}}{2} \frac{C_p \ell F}{KL} \quad , r = \frac{kT}{\ell F} \tag{14}$$

The value of M/r for the parameters used in the previously described calculation is 9.86. These calculations were then repeated with a flux $F = 20{,}750$ joule/(cm^2sec) so as to give a value of $M/r = 4.89$ identical to the one used by Citron. The two dashed curves of Figure 5 are results given by Citron as first and second approximations. The abcissa is a dimensionless time, defined by

$$\tau = \frac{\alpha}{\ell^2} (t - t^*) \tag{15}$$

t^* being the time of front surface melting.

The ordinate is the fraction of the thickness which has melted at dimensionless time τ. The solid curve, labeled $r = 0.235$, depicts the results obtained by the finite element method, and should be in agreement with the results of Citron. The apparent discrepancy at the early time was found to be due to the inapplicability of the Citron method at small values of r, even though r does not appear explicitly in the results. In developing this approximate method, Citron assumed the exponential terms in Equation 12a to be negligible at the melting time, t^*. With this assumption, we have

$$r = \frac{\alpha t^*}{\ell^2} + \frac{1}{3} \tag{16}$$

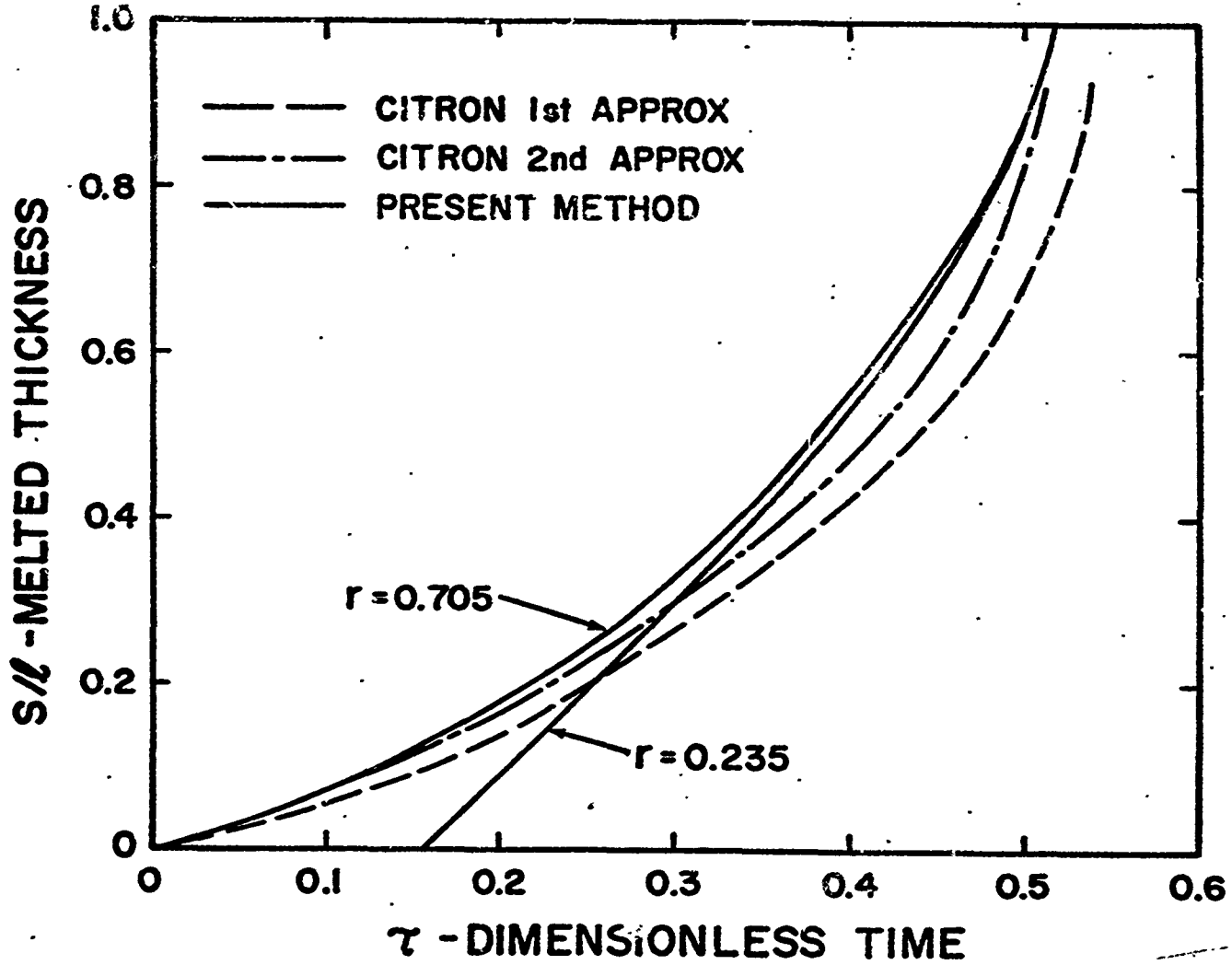

Figure 5 Free Surface Recession Compared for Two Approximate Methods

A r of 0.235 leads to a meaningless negative time of first melting. The approximation made by Citron is valid only for r ~ 2/3 or greater. In order to compare the results of the present method with those given by Citron with a value of r such that his results are valid, calculations were performed for a flux of F = 20,750 joule/(cm^2sec) and all properties those of .3 cm thick aluminum, except that a melting temperature of 2373°K was used. This leads to a value of r of 0.705, with M/r = 4.89. The results of this calculation are given in the Figure as the indicated solid line. Although there is some difference between the two predictions, they are now in qualitative agreement.

c. Two Dimensional Heating of a Thin Sheet

As a third example, we consider the two dimensional temperature distribution resulting from a steady axi-symmetric flux of a Gaussian distribution acting on a thin sheet.

$$F(r) = Fe^{-r^2/2\sigma^2} \qquad (17)$$

The pre-melting solution can be compared with the temperature profiles predicted by a computer program written at AFWL [31] which evaluates the solution given by Olcer [27].

As an example, we will consider a titanium sheet, 0.04 cm. thick and 5 cm. in diameter, initially at 300°K, having thermal properties

ρ = 4.43 gm/cm^3

K = 0.145 joules/(cm°Csec)

C_p = 0.77 joules/(gm°C)

T_m = 1900°K

L = 390 joules/gm

We assume the disk is subjected to a beam of peak intensity of 2000

joules/(sec cm^2) with $\sigma = 0.25$ cm. The pre-melting temperature field was computed using the finite element method by (a) dividing the slab into 10 layers and the radius into 40 annular regions, using a time step of 80 μ sec, and (b) by dividing the sheet into 20 layers and 20 annular rings, using a time step of 20μsec. In both cases, the time step used was about 40% of the minimum value necessary to satisfy the one dimensional stability criterion. In the most extreme case, $\Delta R/\Delta z > 15$, and it was assumed that instabilities would primarily arise due to the axial, rather than the radial, flux. The temperature profiles at $z = 0.002$ cm ($\ell/20$) were found to differ by less than 1% for all r, at times $t = 0.03$ and $t = 0.10$ sec. Temperature profiles as computed with the finite element method (solid lines) are compared with the results of the Fourier series solution (AFWL computer program) in Figure 6. The agreement is excellent, particularly within the beam radius ($2\sigma = 0.5$ cm). Temperature profiles through the slab thickness, at $r = 0.0625$ cm, are compared at two values of time in Table I. The exact solution indicated that melting begins at the front surface at $t = 0.1037$ sec. The two finite element solutions gave 0.1042 and 0.1064 sec., respectively. Although this is a two dimensional problem, axial heat flow predominates and the one dimensional approximation is quite satisfactory for it leads to a predicted time to first melting which is only 10% low.

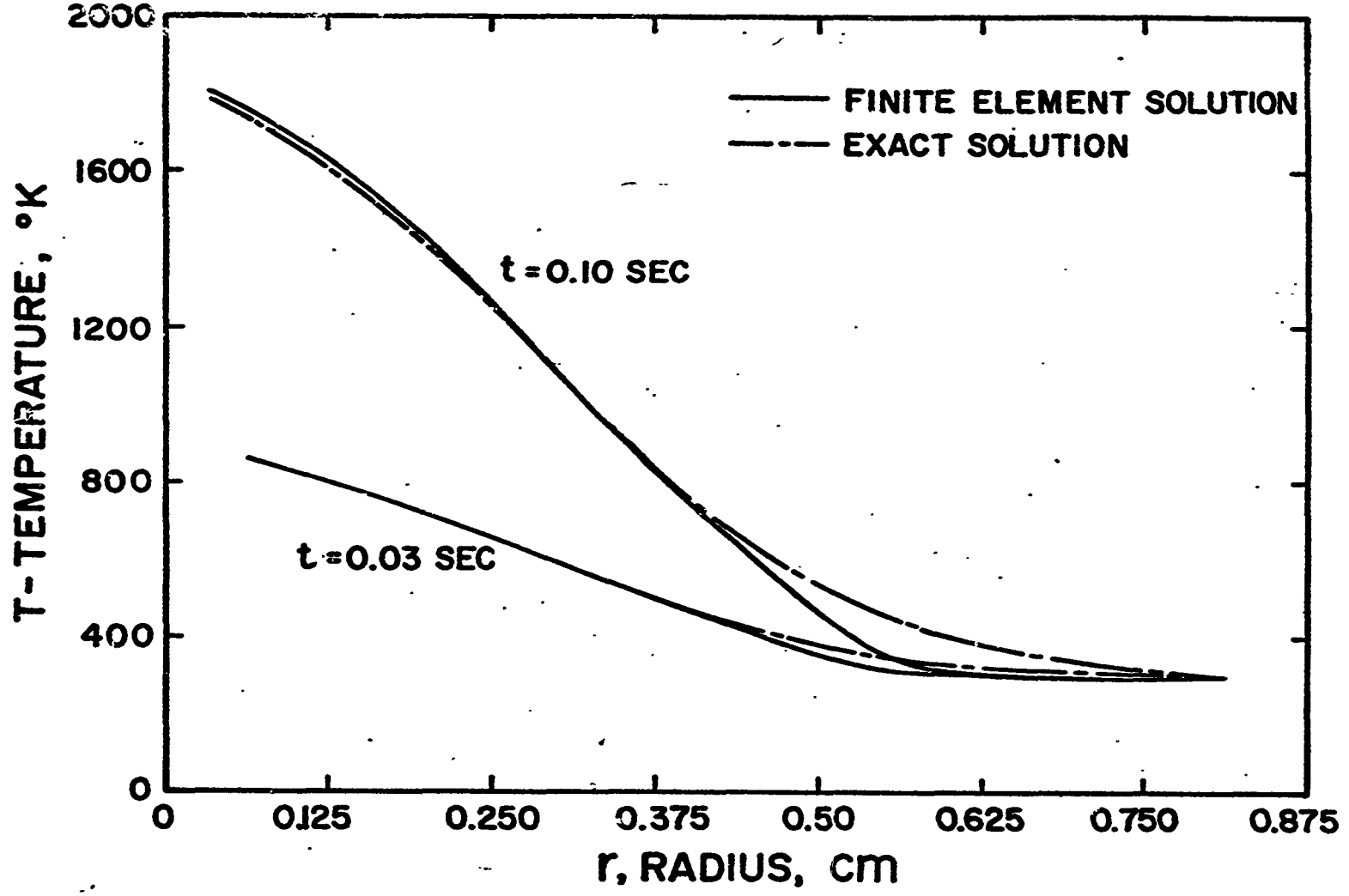

Figure 6 Radial Temperature Distribution at z = 0.002 cm in 0.04 cm Titanium Sheet. F = 2 KW/cm^2, σ = 0.25 cm

Table I

Temperatures (°K) at r = .0625 cm

	t = 0.3 sec			t = .10 sec		
z	Exact Soln	Approximate Soln a	b	Exact Soln	Approximate Soln a	b
.002	852	871	871	1743	1793	1795
.006	807	823	822	1697	1745	1747
.010	766	780	780	1657	1702	1705
.014	730	743	743	1622	1665	1667
.018	699	711	710	1590	1633	1635
.022	673	684	689	1564	1606	1609
.026	652	663	663	1543	1585	1587
.030	637	647	647	1528	1569	1572
.034	626	636	636	1518	1559	1561
.038	621	631	631	1513	1553	1556

IV. APPLICATIONS

In order to gain further experience in the utilization of the new method, several two dimensional heat conduction and melting problems were studied. Applications involving several materials of interest were considered, viz. Al_2O_3, Titanium, Stainless Steel, and Magnesium. In none of these cases is another solution available for comparison, although experimental results corresponding to several of these cases are available.

a. Melting of Thick Alumina Slab

The predicted time of melting through a thick (0.953 cm) slab of Al_2O_3, 5 cm in diameter, due to an absorbed flux of peak intensity of 4000 joules/(cm^2sec) and Gaussian parameter of σ = 0.70 cm was computed with a number of runs. In all cases, the following parameters were assumed

ρ = 3.8 gm/cm^3

L = 1070 joules/gm

K = 0.104 joules/(cm sec °C)

C_p = 0.885 joules/(gm °C)

T_m = 2313 °K

and the initial temperature was taken to be 300°K. The slab was divided into various (10,20, and 40) layers, and the radius into 20 and 40 segments. The time steps which satisfy the one dimensional stability criterion are .147, .0368 and .0092 sec., respectively. The results of the various computations are given as Table II. The temperature at r=a and z=0 at the time of complete melt through was found to have increased only 2°C, hence these results are also applicable to disks of any larger diameter. Excellent agreement between the results of various runs is evident, despite the significant variations in the size of space and time increments. Instantaneous removal of the melted material was assumed.

Table II

Time Required to Melt Al_2O_3 Slab, 0.953 cm Thick and 2.5 cm Diameter with F = 4Kw/cm^2 and σ = 0.70 cm

Run No.	Number Layers	Number Radial Segments	Time Step Sec.	Melt Through Time Sec.
1	10	20	.01	2.86
2	10	40	.01	2.83
3	20	20	.005	2.845
4	20	40	.005	2.85
5	40	40	.0025	2.8475

b. Melting of a Thin Titanium Sheet

We consider now the consequence of a Gaussian beam of $\tau = 0.25$ being absorbed by a 0.04 cm thick sheet of Titanium, initially at 300°K. We take the thermal properties to be as in Section IIIc and $\Delta t = 80$ μ sec. The times at which the various finite segments were found to be completely melted (and assumed to be removed) are depicted graphically in Figure 7. The times given in the Figure are measured from the onset of melting at the front surface. The exact and discretized approximation to the prescribed flux is also shown. The propagation of the free boundary through the sheet is shown as the solid lines in Figure 8 for peak absorbed intensities of 1500, 2000, and 2500 joules/(cm^2sec). For purposes of comparison, the predicted melting rate for one dimensional heating with a peak intensity of 2000 joules/(cm^2sec) is also given as the dashed line. As was noted in Section IIIc a Titanium sheet of this thickness can be quite well approximated by a one dimensional problem, despite the two dimensionality of the temperature distribution, evident from Figure 6. Although no analytical results for this problem are available, the predicted melt through time is in satisfactory agreement with experimental results [31].

c. Melting of Thin Stainless Steel

In an experimental test program conducted at WPAFB, a 16 mil sheet of 304 Stainless Steel was irradiated by a beam having a total power of 9 kilowatts and a diameter of 2.44 cm. Temperatures were measured by a thermocouple attached to the rear surface. The results [31] were as indicated by the circles in Figure 9. From motion picture films taken of the event, the time at which complete melt through occured was estimated to be 0.40 sec. Assuming a Gaussian beam profile, a peak intensity of 3850 joules/(cm^2sec) was estimated from the relationship

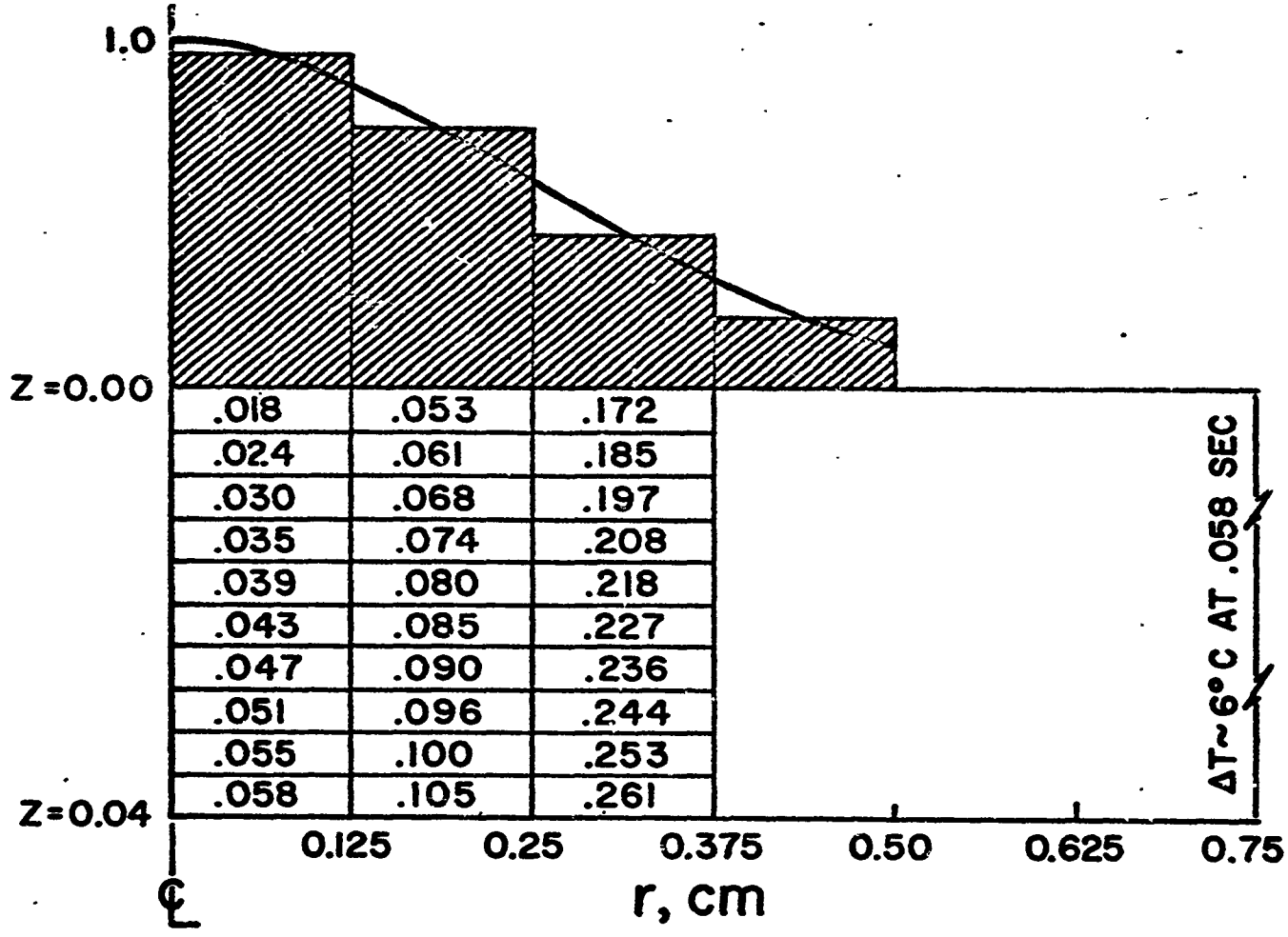

Figure 7 Mass Removal Times for Melting of Thin Titanium Sheet, F = 2 KW/cm^2, σ = .25 cm

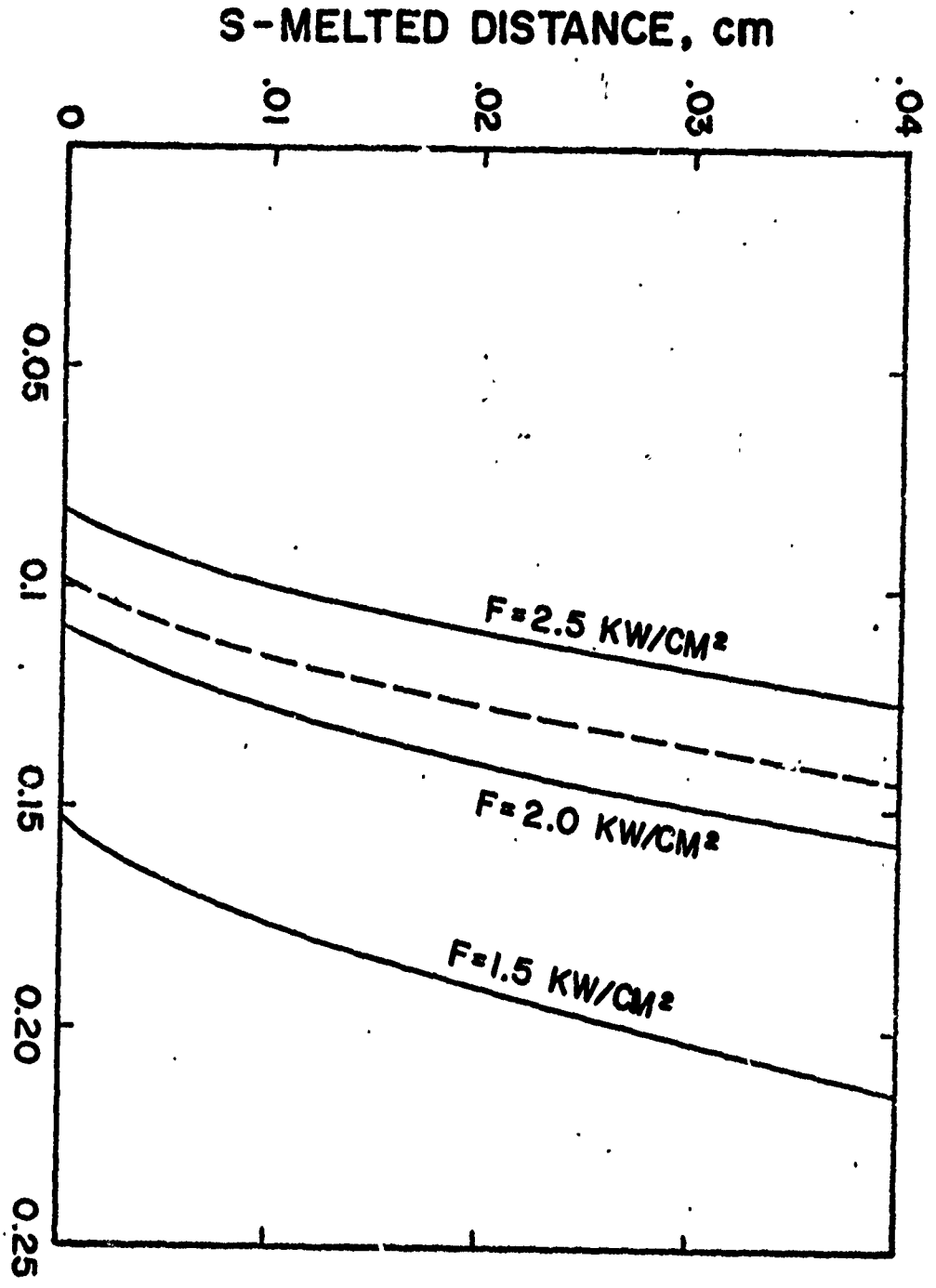

Figure 8 Location of Free Surface in Melting Titanium Sheet for Various Flux Levels

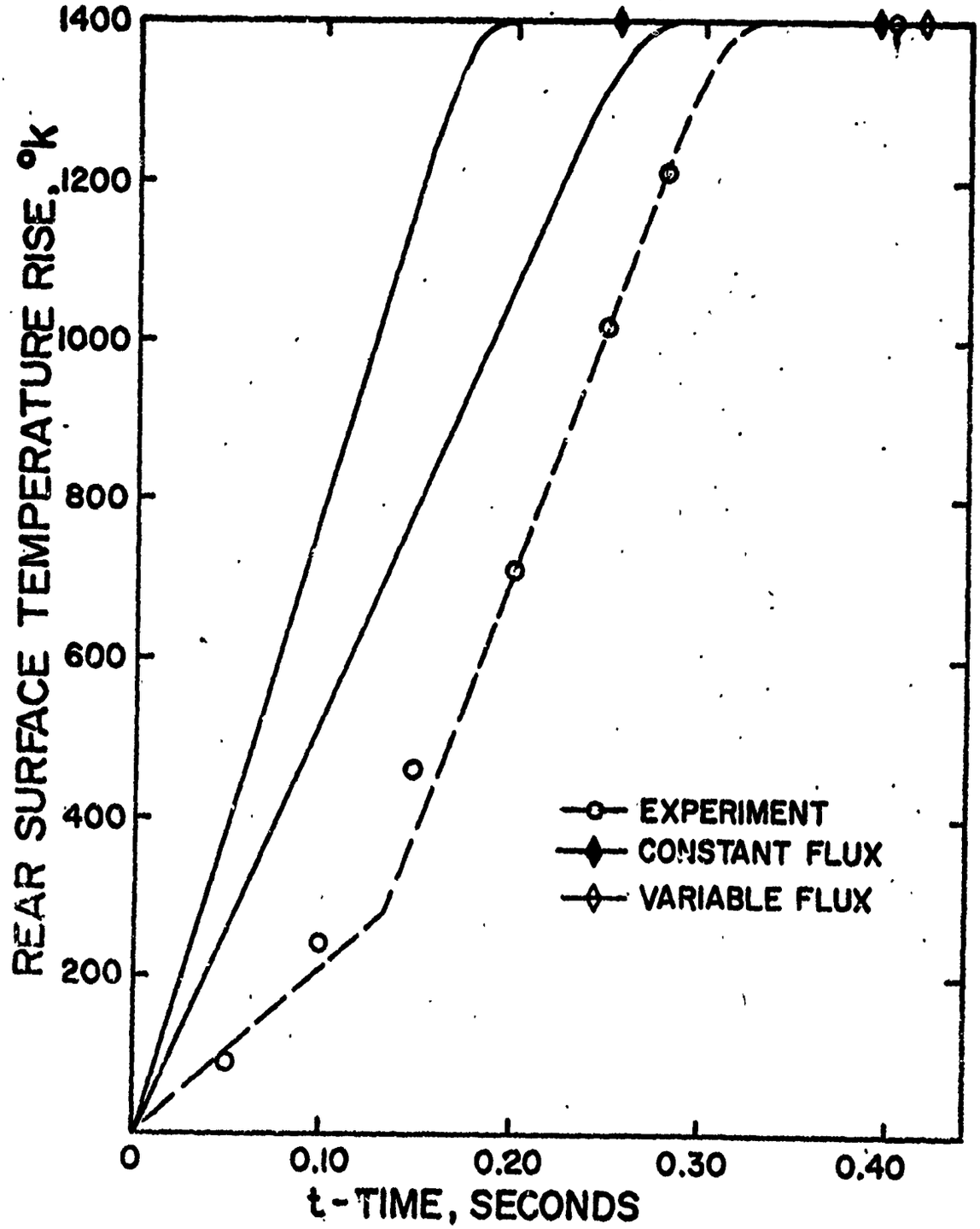

Figure 9 Rear Surface Temperatures in Melting 304 Stainless Steel at Various Flux Levels

$$\mathcal{P} = 2\pi F_0^2 \qquad (17)$$

The material was assumed to be removed immediately after melting. Calculations of the temperature distribution and progression of melting were made, using the following thermal properties:

$\rho = 7.9$ gm/cm^3

K = 0.24 joules/(sec cm °C)

$C_p = 0.42$ joules/(gm°C)

L = 390 joules/gm

$T_m = 1700$°K

An initial temperature of 300°K was assumed and a disk radius of 5 cm was used. The disk was divided into 10 layers and 20 annular rings, and a time step of 50μsec was used in the calculations. Absorbtances of 0.2 and 0.3 were first assumed, leading to the predicted rear surface temperatures shown as solid lines in Figure 9. The predicted melt through times for these absorbtances were 0.387 sec and 0.256 sec, respectively. While an absorbtance of 0.2 leads to a melting time which is close to the observed value, the predicted temperature history at the rear surface is significantly different from that which was observed. The observations appear to correspond to an absorbtance of about 0.08 at early times, followed by a transition to a higher value, perhaps 0.24. Based on this observation, the computer program was modified so that the flux entering a cell on the surface would increase from the lower to the higher value when the average temperature of that cell reached 600°K. The resulting predicted time for complete melt through is 0.416 sec., and the thermal history of the rear surface can be seen from the dashed curve to be in satisfactory agreement with the experimentally determined results. These results would seem to indicate that a significant change in absorbtance of Stainless Steel takes place

at temperatures well below the melting temperature. It is conjectured that this change takes place due to the formation of an oxide layer on the heated surface. The formation of such an oxide layer can be expected to depend not only on the current temperature, but also on the temperature history of the surface, which would be dependent on the magnitude of the applied flux.

d. Melting of Magnesium Sheets

Since the time required for complete melt through of a sheet is very easy to calculate (Equation 13) if the heat flow is entirely axial, and very difficult if it is not, it would be very advantageous if there were some means of knowing, a priori, whether or not a one dimensional approximation is appropriate for any given heating situation.

A computational study, considering magnesium of thicknesses from 0.08 cm to 1.28 cm, beam parameters (σ) from .32 and 1.28 cm and peak intensities of 1, 3, 5, and 10^k joules/(cm^2sec) was undertaken. A cylindrical disk, 10 cm. in diameter, and having the following thermal properties was considered, and the melt was assumed to be instantaneously removed

$\rho = 1.77$ gm/cm^3

$K = 0.96$ joules/(cm sec °C)

$C_p = 1.04$ joule/(gm °C)

$L = 338$ joules/gm

$T_m = 905$°K

Each disk was subdivided into 10 layers, regardless of thickness, and the radius was divided into 20 segments, except in the case of $\sigma = 0.32$. The discretization of the incident flux was found to lead to significant errors when the parameter σ is comparable to the size of the radial segment, or

greater, necessitating the use of 40 segments for the case of $\sigma = 0.32$ cm. In all cases, time steps were kept below 50% of the minimum value required to insure stability.

The results of these calculations are given as Figure 10, where the time to melt is given as a function of thickness for various beam parameters. As can be seen from the figure, large spot size and higher levels of flux both lead to melting times in closer agreement with the melting time for one dimensional problems, as given by Equation 13.

It might be expected that the ratio σ/ℓ, being the ratio of the only two lengths present, would characterize the "degree of one-dimensionality" of a given heating situation for a given material. This was not found to be the case. The ratio, for example, of t/t_1 for $F = 5$ Kw/cm^2 can be seen to vary from 1.348 at $\sigma = \ell = 0.32$ down to less than 1.1 for $\sigma = \ell = 1.28$. A more promising means of interpretation appears to be a plot such as is given in Figure 11, wherein the normalized melting time is given as a function of total power, for various sheet thicknesses. The time to melt through has been normalized by division by the time required for a uniform beam of the same peak intensity to melt through a similar slab, assuming one dimensional heat flow. Computed points are shown, for the various thicknesses, as circles, squares and triangles, but all the computed points do not appear to fall on smooth curves. For this reason, deductions about the one-dimensionality of problems well outside the range of parameters considered in this study should not be attempted, but these results do appear to suggest that the total power is a more significant parameter than the peak intensity, spot size, or the ratio of spot size to thickness. From Figure 11, it can be seen that the curves of t/t_1,

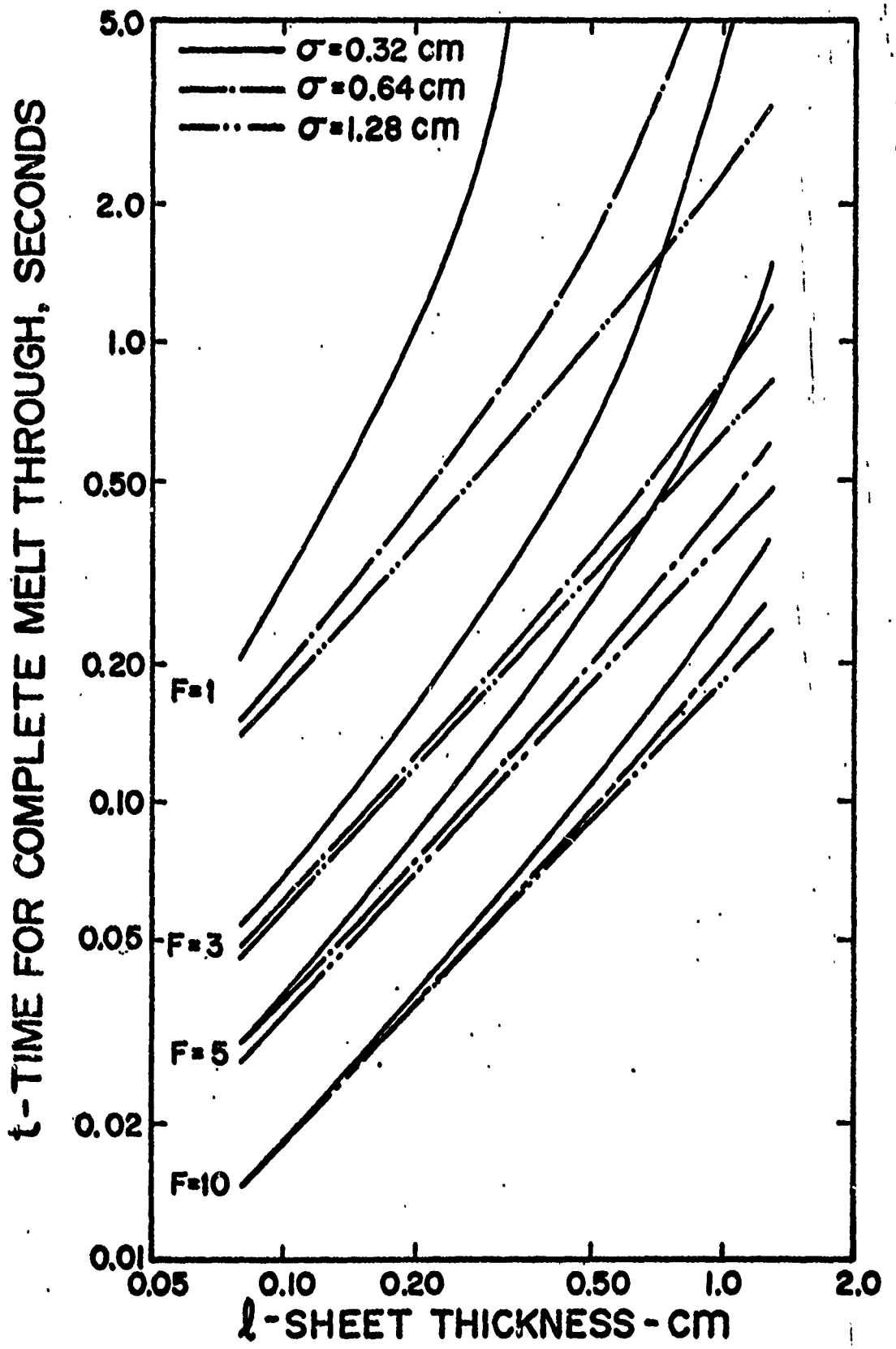

Figure 10 Melting Time of Magnesium Sheet as a Function of Thickness, for Various Beam Parameters

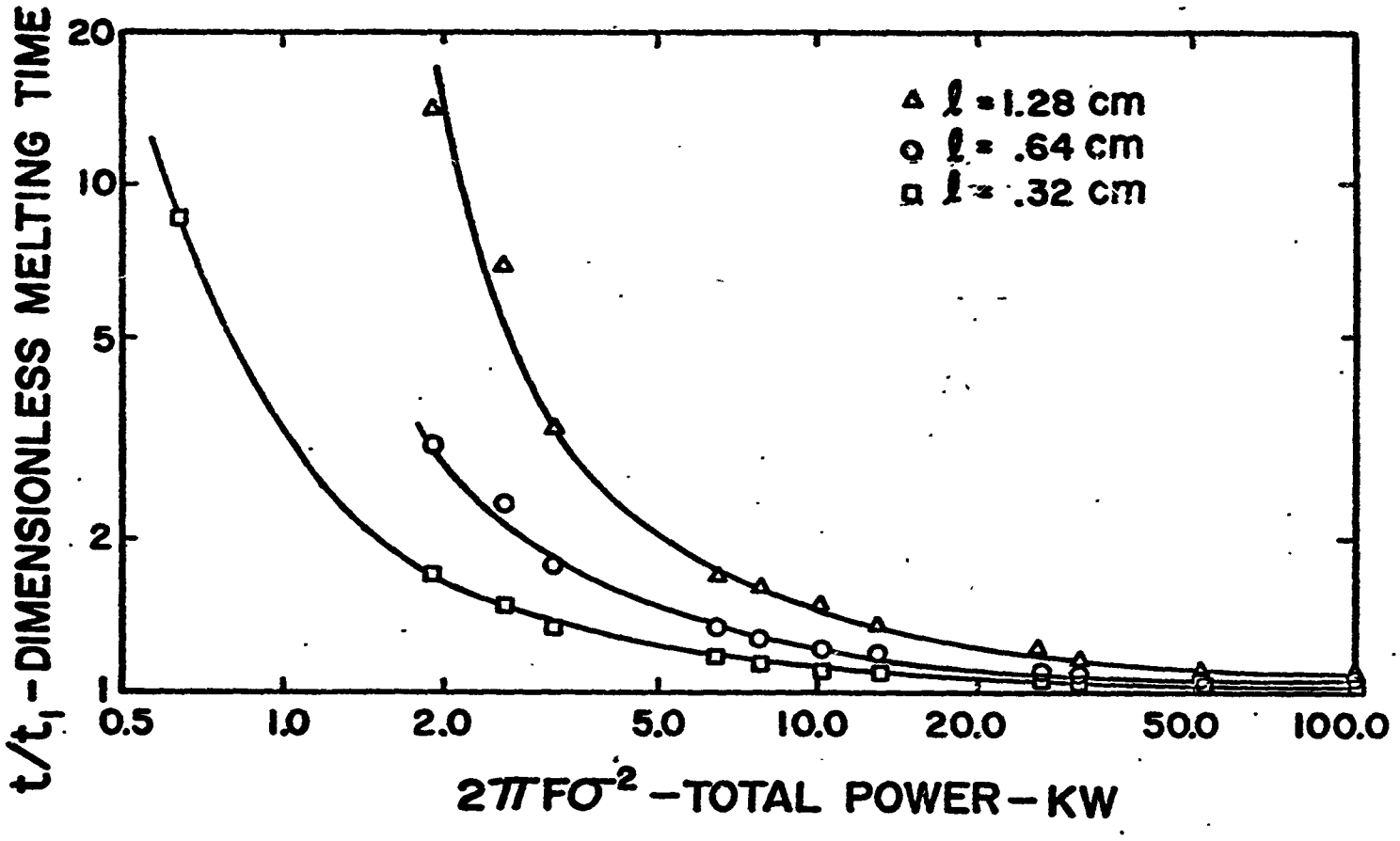

Figure 11 Dimensionless Melting Time for Magnesium as a Function of Total Power, for Various Thicknesses

vs. power for different thicknesses are all parallel, that is, if each is translated along the horizontal axis, the three curves can be made to concide. Moreover, the distance that each curve must be translated is inversely proportional to ℓ, i.e., $t/t_1 = f(P/\ell)$. Thus, rather than a dependence of melting time on the various parameters having the form

$$t_m = f(\sigma, F, \ell) \tag{18}$$

we can write

$$t_m = t_1 f(P/\ell) \tag{19}$$

where t_m is the time required to melt a small hole with a beam of intensity F, gaussian parameter σ in a thickness ℓ of some material

t_1 is the time required for one dimensional melting of the same material, with flux F and thickness ℓ,

$$t_1 = [L + C_p(T_m - T_o)] \frac{\rho \ell}{F} \tag{20}$$

P is the total power

$$P = 2\pi F \sigma^2 \tag{21}$$

ℓ is the thickness

and f is a single valued function, which we may expect to be different for each material.

For magnesium, f(40,000) ~ 1.1 so the time required for melting can be approximated to within 10% by the time required for one dimension of melting whenever $2\pi F\sigma^2/\ell > 40{,}000$ joules/(cm sec). For comparison, f(4500) ~ 2.0 indicating that melting will required twice as long as in the one dimensional care. In this case, 50% of the flux applied at the center of the heated area is conducted axially, and 50% radially.

The results presented in Figures 10 and 11 should be regarded as being somewhat tentative, as multiple computations of each case, using various spatial and temporal step sizes, were not undertaken to insure convergence of these numerical results.

V. CONCLUSION

In the preceding sections, a method for determining numerically the temperature distribution within a disk of arbitrary dimensions and thermal properties subject to a prescribed flux over one face has been given. The method takes into account a change of phase, and can be used to predict the time required for partial or complete melting of the material. The method was applied to a number of examples and was found to give satisfactory agreement with other solutions and with experimental evidence when available.

Although no calculations for composite materials were included in the examples, the method was developed with such calculations in mind, specifically, so that one or more layers could be taken to have the thermal properties of a material which might be used as a protective coating.

The method can also be expected to handle a number of other problems, and can be modified to incorporate other factors. A single phase change from solid to vapor (rather than liquid) can be handled by the method without any modification. Two phase changes, as first a transition from solid to liquid, followed by heating of the retained liquid and the eventual vaporization of the liquid can be readily incorporated into the method. Provision was also made for the addition of heat to each cell, so that heat releasing reactions within the material might be considered. A time dependent prescribed flux can be handled without difficulty, as can boundary conditions other than the completely insulated boundary which was assumed in the examples considered here. Changes in flux as a function of surface temperature, such as can occur through a temperature dependent absorbtance can be treated by this method, as was demonstrated in one example. Prescribed fluxes which are not axially symmetric could

in principle be treated by a finite element method such as this, but it is anticipated that the large number of elements which would be necessary to handle a third component in the heat flux vector would require inordinately large computer storage and running times for these transient applications. Certain quasi-three dimensional problems, in which the prescribed flux is a two dimensional function of the surface coordinates, but for which the heat flow is primarily one dimensional are probably amenable, however. Such problems might arise from the spatial and temporal fluctuation of a distribution of radiation.

Significant remaining problems in determining the consequences of laser heating of materials would appear to be in the development of models for predicting the rate of mass removal and for predicting the rate at which heat is liberated in such materials as titanium and magnesium, as has been observed, and in determining the absorbtance as a function of surface conditions.

ACKNOWLEDGEMENT

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APPENDIX

A Fortran Program was written to perform the two dimensional heating and melting calculations by the method described in Section II. A brief documentation follows:

Card Numbers	Computation Performed
15-20	Input data read in. All temperatures Kelvin; other dimensions joules, cm, gm, and sec. NC and NR are each one greater than the number of columns and rows. A is the radius and D the thickness, ΔT the time step and TREF the initial temperature. FL is the incident flux, and ABC the fraction absorbed.
25-45	Arrays, describing the properties of the elements are computed. Here, a homogeneous material is assumed.
48-56	Gaussian beam profile approximated by discretization. For small SIG (σ) only those cells entirely within the beam receive flux, leading to significant error if DR comparable to SIG.
60	Beginning of main computational cycle.
66-72	Temperature distribution printed every NPRO time steps, if desired (NPUT = 0). Temperatures not printed for other value of NPUT.
76-82	Heat flux for all but edge rows and columns.
83-94	Heat flux for all remaining cells computed.
95-106	Heat balance performed.

Card Numbers	Computation Performed
111-112	Raise temperature of cell, if not at melting.
114-115	Apply heat towards the phase change, if cell is at melting temperature.
121-129	If cell completely melted, remove from problem.
134	End calculation if melt through has occurred at disk center.
135	End of main computational cycle.
137-144	Print parameters used for computation.
145	Give error count (number of times the heat flux "went the wrong way") as determined by counters in lines 79 and 86.
147-153	Print time at which each cell was removed.

```
      PROGRAM SUPER(INPUT,OUTPUT)
C     MELTER DECK INCORPORATING ALL CHANGES
C       SUPER DECK DUPLICATED ON 18 SEPTEMBER 1971
      COMMON    HEAT(20,20),CONF(20,20),CONZ(20,20),FLUX(21,20),
     1CAPP(20,20),T(21,21),THET(20,20),DZ(20,20),DR(20,20),DELS(20,20),
     2GONE(20,20),MAX(10),FLUY2(21,20)
      DATA PI/3.1415/
      NPRO = 50
      NPRO = 100
      NPUT = 1
      NPUT = 1
      NPUT = 2
C     0 FOR EVERYTHING  1 FOR MELTING TIMES 2 FOR NOTHING
  999 CONTINUE
      READ 901, RHO,FUSION,CON,CP,TMLT
  901 FORMAT(5F10.4)
       READ 902, A,B,NC,NR,DT,NSTEP
  902 FORMAT(2F10.4,2I10,F10.7,I10)
      READ 903,TREF,ABC,FL,SIG
  903 FORMAT(4F10.4)
      IF(NPUT.NE.2) PRINT 101
  101 FORMAT(1H1)
      DO 5 MI = 1,10
    5 MAX(MI) = 0
      COND=CON
      CAP=CON/(RHO*CP)
      ENR = NR
      FNC=NC
      DR=A/(FNC-1.0)
      DZ=B/(FNR-1.0)
      NCR=NR-1
      NCC=NC-1
      DO 20 I=1,NCC
      FI=I
      RI=FI*DR-DR
      RO=RI+DR
      AREA=PI*(RO*RO-RI*RI)
      DO 20 J=1,NCR
      CONF(J,I) = 2.0
```

```
      HEAT(J,I)=FUSION*RHO*AREA*DZ
      CONP(J,I)=CON*2.0*PI*RC*DZ
      CONZ(J,I)=CON*AREA
      FLUX(J,I)=0.0
      CAPP(J,I)=AREA*DZ*CP*RHO
      THET(J,I) = TREF
   20 CONTINUE
C     SUPPLY EXTERNAL FLUX
      IF(SIG.LE.A/2.) GO TO 123
      N2SIG=NCC
      GO TO 124
  123 N2SIG=2.0*SIG/DR
  124 DO 125 I=1,N2SIG
      FI=I
      R=(FI-.5)*DR
  125 FLUX(1,I)=FL*EXP(-.5*(R/SIG)**2)*PI*(FI*FI-(FI-1.)*(FI-1.))*DP*DR
     1*ABC
C     BEGIN TIME SEQUENCE
      TIME=0.0
      KX = NPRO - 1
      DO 200 K=1,NSTEP
      KX=KX+1
      IF(KX.NE.NPRC) GO TO 25
      IF(NPUT.GE.1) GO TO 25
      PRINT 23,TIME
      KX=0
      DO 21 KK=1,NCR
      NCP = NCC
      IF(NCC.GT.20) NCP = 20
      PRINT 22,(THET(KK,JJ), JJ=1,NCP)
   22 FORMAT(20F6.0)
   23 FORMAT (/10H TEMPS AT ,E12.4,7HSECONDS)
   21 CONTINUE
   25 CONTINUE
      M=NCC-1
      N=NCP-1
      DO 45 I=1,N
      DO 40 J=1,M
      QZ(I,J)=(THET(I,J)-THET(I+1,J))*CONZ(I,J)/DZ
      IF(QZ(I,J).LT.0.0) MAX(3) = MAX(3) + 1
```

```
      DO 45 I = 1,N
      J = NCC
      QZ(I,NCC) = (THET(I,J)-THET(I+1,J))*CONZ(I,J)/DZ
      IF(QZ(I,J).LT.0.0) MAY(3) = MAY(3) + 1
   50 QR(I,NCC)=0.0
   55 CONTINUE
      DO 65 I = 1,M
      QR(NCR,I) =(THET(NCR,I) - THET(NCR,I+1))*CONR(NCR,I)/DR
   60 QZ(NCR,I)=0.0
   65 CONTINUE
      QZ(NCR,NCC) = 0.0
      QR(NCR,NCC) = 0.0
      DO 75 I=2,NCR
      DO 70 J=2,NCC
      DELQ(I,J)=FLUX(I,J)+QZ(I-1,J)+QR(I,J-1)
     1 -QR(I,J)-QZ(I,J)
   70 CONTINUE
   75 CONTINUE
      DO 95 J=2,NCC
   90 DELQ(1,J)=FLUX(1,J)-QZ(1,J)-QR(1,J)+QR(1,J-1)
   95 CONTINUE
      DO  105 I=2,NCR
  100 DELQ(I,1)=FLUX(I,1)-QZ(I,1)-QR(I,1)+QZ(I-1,1)
  105 CONTINUE
      DELQ(1,1)=FLUX(1,1)-QZ(1,1)-QR(1,1)
      DO  115 I=1,NCR
      DO 110 J=1,NCC
      IF (CONZ(I,J).EQ.0.0) GO TO 110
      IF(THET(I,J).GE.TMLT) GO TO 90
      THET(I,J)=THET(I,J)+DELQ(I,J)*DT/CAPP(I,J)
      GO TO 110
   90 HEAT(I,J)=HEAT(I,J)-DELQ(I,J)*DT
  115 HEAT(I,J)=HEAT(I,J)-(THET(I,J)-TMLT)*CAPP(I,J)
      IF(THET(I,J).EQ.TMLT) GO TO 82
      THET(I,J)=TMLT
      IF(NPRT.GE.2) GO TO 82
      PRINT 81,TIME,I,J
```

```
120          42 CONTINUE
                IF(HEAT(I,J).LE.0.0)GO TO 120
                GO TO 110
             C     MODIFY HERE FOR MELTING MATERIAL
               120 CONZ(I,J)=0.0
125             CONR(I,J)=0.0
                FLUX(I+1,J)=FLUX(I,J)
                FLUX(I,J)=0.0
                GONE(I,J) = TIME
                THET(I,J) = 0.0
130            110 CONTINUE
               115 CONTINUE
                81 FORMAT(6H TIME=,E12.4,2I5,13H CELL MELTING)
             C     END CALCULATION AT TIME STEP
                IF(THET(NCP,1).EQ.0.0) GO TO 201
135            200 TIME=TIME+DT
               201 CONTINUE
                PRINT 801, PHC,FUSION,CON,CP,TMLT
               801 FORMAT(*1PC=,*F10.4,*FUSION=*,F10.4,*CON=*,F10.4,*CP=*,F10.4,*TMLT
                  1=*,F10.4)
140             PRINT 802, A,B,NC,NR,DT,NSTEP
               802 FORMAT(* A=*,F10.4,*B=*,F10.4,*NC=*,I5,*NR=*,I5,*DT=*,F10.6
                  1,*NSTEP=*,I5)
                PRINT 803, TREF,ABC,FL,SIG
               803 FORMAT(* TREF=*,F10.4,*ABC=*,F10.4,*FL=*,F10.4,*SIG=*,F10.4)
145             PRINT 860,MAX
               860 FORMAT (* MAX*,10I10)
                PRINT 301
               301 FORMAT(//16H TIME OF MELTING)
                DO 400 I = 1,NCP
150             NCP = NCC
                IF(NCC.GT.20) NCP = 20
               400 PRINT401,(GONE(I,J),J = 1,NCP)
              401  FORMAT(1X,5F8.4,15F5.2 )
                GO TO 999
155             STOP
                END
```

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